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Brownian motion of electrons

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Abstract. The concepts of Chandrasekhar are used to derive a new theory which describes the random motion of a particle less massive than the fluid particles. A low-density Lorentz gas with isotropic scattering is used as a model. Probability densities are calculated and the mean square displacement is found to agree with Ornstein's result,

$$\langle x^2 \rangle = 2D\beta^{-1}(\beta t - 1 + e^{-\beta t})$$

only if all particles have the same collision rate. An expression is given for the displacement in a magnetic field, which is valid at all times; in the limit of $t \to \infty$ the usual diffusion constant is obtained. The theory is applied to the scattering of radiation by electrons in a plasma, and the spectra are found to be narrowed by collisions, although features such as the peaks at the plasma frequency or multiples of the cyclotron frequency are broadened. The collision term used in the linearized Boltzmann equation is more realistic than the widely used BGK term, and the collision rate can vary with speed.

1. Introduction

The theory of Brownian motion has often been used to describe the motion of particles in a plasma. However, it has been applied indiscriminately to cover particles with any mass, whatever the mechanism for their interaction with the plasma. In this paper the concepts of Chandrasekhar (1943) are used to derive a new theory of Brownian motion. It provides an accurate description of the motion of electrons when collisions with neutral atoms are dominant. The basic theory is developed in § 2 and the probability density for the displacement of an electron is obtained in the following section. Section 4 compares the mean square displacement with the result obtained classically. Next, the theory is generalized to include a magnetic field. In § 6 the theory is applied to the scattering of electromagnetic radiation by the electrons in a plasma, and then the modification of the spectra by collective effects is calculated. Section 8 compares the spectra, probability densities and mean square displacements with the results of other theories, for example the Fokker–Planck equation and the usual form of the Boltzmann equation. Finally, the application of this theory to real plasmas and to other systems is discussed.

Classically (Einstein 1956) a particle, which is much more massive than the fluid molecules, undergoes Brownian motion because of the cumulative effect of many small impacts from the molecules. Over time intervals longer than the mean collision time the motion is described by the Langevin equation (Chandrasekhar 1943)

$$\frac{dv}{dt} = -\beta v + A(t). \tag{1}$$

Here the first term on the right represents the dynamical friction with the fluid, and the second a stochastic acceleration which is assumed to have a Gaussian distribution. The mean square displacement (Uhlenbeck and Ornstein 1930) in the x direction is

$$\langle x^2 \rangle = 2D\beta^{-1}(\beta t - 1 + e^{-\beta t}) \tag{2}$$

where the diffusion constant D is $\kappa T/m\beta$ for a fluid temperature T and a particle of mass m. When $\beta t \gg 1$, Einstein's result $\langle x^2 \rangle = 2Dt$ is valid and the probability density of the net displacement, $W(x;t)$, is governed by the diffusion equation.

An electron in a gas also travels in an irregular manner. Here, contrary to the usual Brownian motion, each impact has a profound effect on the velocity of the electron, although the transfer of energy between the electron and the gas is slow. The essential

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features of the electronic Brownian motion are maintained if we neglect this transfer entirely. Thus inelastic collisions are ignored and the gas atoms are assumed to have infinite mass. An electron is usually deviated through an angle of less than 90° when it bounces off a gas atom. However, outside the region around 0° the asymmetry is usually small, and isotropic scattering is a good approximation if the momentum transfer collision rate γ (Allis 1956) is used in all calculations. The theory developed below can handle any functional dependence of the collision rate on the electron speed c, but a constant mean free path λ will usually be adopted, i.e. $\gamma(c) = c/\lambda$. This is appropriate for the model we are using—the motion of a particle in a Lorentz gas of hard spheres. Measurements (Adler and Margenau 1950) of the electrical conductivity of a weakly ionized plasma have shown that constant λ is a more realistic assumption than is constant γ.

2. Brownian motion of an electron

Because the scattering is isotropic, the velocities before and after a collision are independent, except that the speed must remain constant. Hence, for an electron with speed c, the x component of velocity, v, has a rectangular probability distribution:

$$\begin{aligned} f(v|c) &= \tfrac{1}{2}c^{-1} \text{ for } |v| \leqslant c \\ &= 0 \text{ otherwise.} \end{aligned}$$

The probability that the electron travels a net distance x in a time t is denoted by $W(x; t)$. The electron has a chance $\mathrm{e}^{-\gamma t}$ of moving to x without colliding on the way; hence W has a contribution given by

$$W_0(x; t)\,\mathrm{d}x = \mathrm{e}^{-\gamma t} f(v|c)\,\mathrm{d}v$$

subject to $x = vt$. The probability that the electron reaches x' and then makes its first collision during $\mathrm{d}t'$ is $W_0(x'; t')\gamma\,\mathrm{d}t'$, and that it then continues so as to reach x at t is $W(x-x'; t-t')$. These probabilities are multiplied and then integrated over x' and t' to yield a recursive equation for W:

$$W(x; t) = W_0(x; t) + \gamma \int_0^t \int_{-\infty}^{\infty} W_0(x'; t') W(x-x'; t-t')\,\mathrm{d}x'\,\mathrm{d}t'.$$

If we introduce the spectrum

$$S(k, \omega) = \int_0^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}x \exp(\mathrm{i}kx - \mathrm{i}\omega t) W(x; t) \tag{3}$$

with a similar definition for S_0 and use the convolution theorem, the integral equation yields

$$\begin{aligned} S(k, \omega) &= S_0 + \gamma S_0 S \\ &= S_0(1-\gamma S_0)^{-1} \end{aligned} \tag{4}$$

where

$$\begin{aligned} S_0(k, \omega) &= \int_0^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}v \exp(\mathrm{i}kvt - \mathrm{i}\omega t - \gamma t) f(v|c) \\ &= \int_{-\infty}^{\infty} \frac{f(v|c)\,\mathrm{d}v}{\gamma + \mathrm{i}\omega - \mathrm{i}kv} \\ &= \frac{\mathrm{i}}{2ck} \ln\left(\frac{\gamma + \mathrm{i}\omega - \mathrm{i}ck}{\gamma + \mathrm{i}\omega + \mathrm{i}ck}\right). \end{aligned} \tag{5}$$

Figure 1 shows the real part of S as a function of ω for various values of λ. The imaginary part is an antisymmetric function of real ω.

In works on classical Brownian theory the original velocity is often left as a parameter in the expression for the probability density. Here this velocity has an ephemeral existence, but the transform of the density $W(x;\, t|v_0)$ can be calculated easily. The first path has a

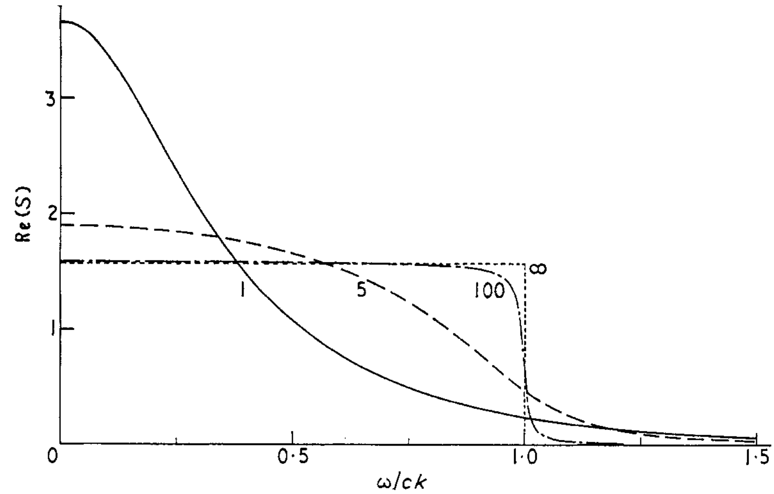

Figure 1. Spectra $S(k,\, \omega)$ for monoenergetic electrons with mean free path λ given by $\lambda k\, =\, 1,\, 5,\, 100$ and ∞.

probability $e^{-\gamma t}$ of lasting for a time t, by which time the electron will have reached $v_0 t$, so that the transform of $W_0(x;\, t|v_0)$ is

$$\int_0^\infty \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}x \exp(\mathrm{i}kx - \mathrm{i}\omega t - \gamma t)\delta(x - v_0 t) = (\gamma + \mathrm{i}\omega - \mathrm{i}kv_0)^{-1}.$$

All the succeeding paths are independent of v_0, so that the complete spectrum is

$$S(k,\, \omega|v_0) = (\gamma + \mathrm{i}\omega - \mathrm{i}kv_0)^{-1}(1 - \gamma S_0)^{-1}. \tag{6}$$

To obtain the spectrum for a group of electrons with a Maxwellian velocity distribution, an average must be taken over c. In obtaining the curves shown in figure 2 a constant mean

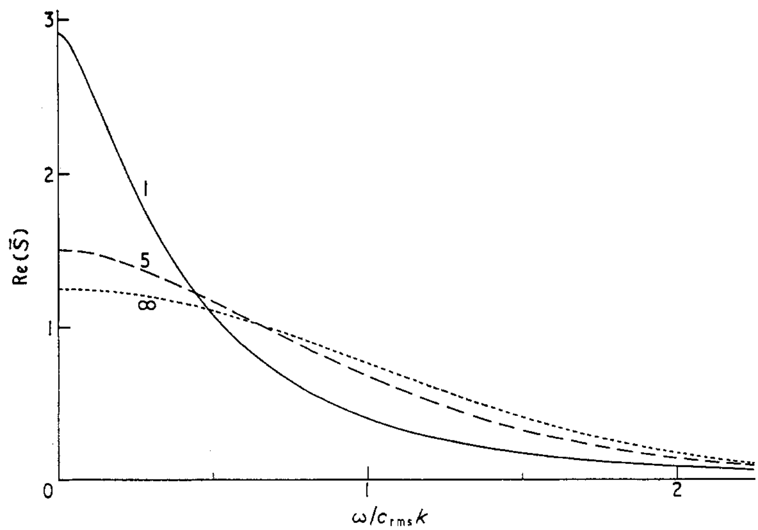

Figure 2. Spectra $S(k,\, \omega)$ for electrons with a Maxwellian velocity distribution and constant mean free path λ given by $\lambda k\, =\, 1,\, 5$ and ∞.

free path was assumed. Thus the averaged spectrum is

$$\bar{S}(k, \omega) = \int S(k, \omega) f_0(c)\, \mathrm{d}c \tag{7}$$

where

$$f_0(c) = c^2 \left(\frac{m}{\kappa T}\right)^{3/2} \left(\frac{2}{\pi}\right)^{1/2} \exp\left(\frac{-mc^2}{2\kappa T}\right)$$

and

$$\gamma(c) = \frac{c}{\lambda}.$$

3. Probability density

The inverse Laplace transform of the spectrum $S(k, \omega)$ is

$$\chi(k, t) = (2\pi)^{-1} \int S(k, \omega)\, \mathrm{e}^{\mathrm{i}\omega t}\, \mathrm{d}\omega$$

where the integration is along a line parallel to the real axis and below all the singularities of the integrand. There is a branch cut between $\omega = \mathrm{i}\gamma + ck$ and $\omega = \mathrm{i}\gamma - ck$, and for real k there may be a pole on the positive imaginary axis, so that the integration can be made along the real axis. Contour integration around these singularities did not give any simplification, so the integral was computed numerically. When k is real, χ is a real symmetric function of k and it can be expressed as

$$\chi(k, t) = \pi^{-1} \int_0^\infty \mathrm{Re}(S) \cos \omega t\, \mathrm{d}\omega - \pi^{-1} \int_0^\infty \mathrm{Im}(S) \sin \omega t\, \mathrm{d}\omega. \tag{8}$$

The first integral here is easy to compute, but the second is difficult because $\mathrm{Im}(S)$ is $\mathrm{O}(\omega^{-1})$ when ω is large. However, we note that the Laplace transformation in equation (3) is equivalent to a Fourier transformation if W and χ are defined to be zero when $t < 0$. Adding $\chi(k, -t) = 0$ to equation (8) gives

$$\chi(k, t) = 2\pi^{-1} \int_0^\infty \mathrm{Re}(S) \cos \omega t\, \mathrm{d}\omega. \tag{9}$$

Finally, the probability density is found using

$$W(x; t) = \pi^{-1} \int_0^\infty \chi(k, t) \cos kx\, \mathrm{d}k. \tag{10}$$

For a group of electrons having a Maxwellian velocity distribution and constant mean free path the function χ is replaced by

$$\bar{\chi}(k, t) = 2\pi^{-1} \int_0^\infty \mathrm{d}\omega \int_0^\infty \mathrm{d}c\, \mathrm{Re}(S) \cos \omega t\, f_0(c).$$

Changing the variable ω to ξc,

$$\bar{\chi}(k, t) = \frac{2}{\pi\lambda} \int_0^\infty \mathrm{Re}\left(\frac{\gamma S_0}{1 - \gamma S_0}\right) Z(\xi t)\, \mathrm{d}\xi \tag{11}$$

where

$$\gamma S_0 = \frac{\mathrm{i}}{2k\lambda} \ln\left(\frac{\xi - k - \mathrm{i}/\lambda}{\xi + k - \mathrm{i}/\lambda}\right) \tag{12}$$

and

$$Z(\xi t) = \int f_0(c)\cos(\xi tc)\,\mathrm{d}c$$

$$= \left(1 - \frac{\xi^2 t^2 \kappa T}{m}\right)\exp\left(\frac{-\frac{1}{2}\xi^2 t^2 \kappa T}{m}\right). \qquad (13)$$

Figure 3 shows the resulting probability density $\bar{W}(x; t)$ for three values of t.

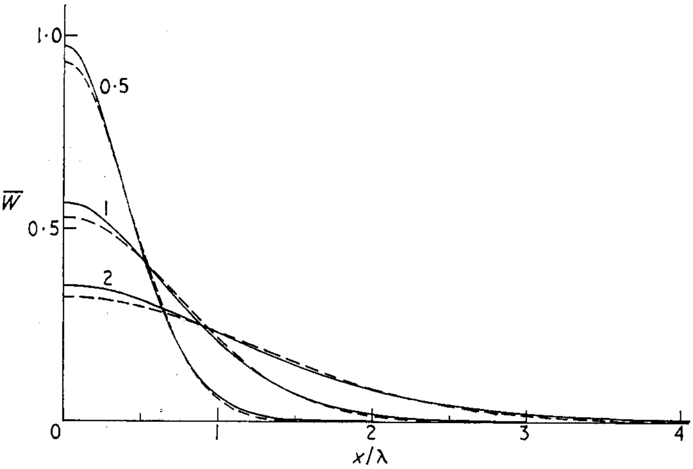

Figure 3. Probability density $\bar{W}(x; t)$ for electrons with a Maxwellian velocity distribution and constant mean free path when $\lambda^{-1}c_{\mathrm{rms}}t = 0{\cdot}5$, 1 and 2. The broken lines show Gaussian distributions having the same variances.

In addition to the one-dimensional density $W(x; t)$, there is the three-dimensional form $\rho(\boldsymbol{r}; t)$. This is not merely a product of three W functions because the displacements in different directions are correlated. However, the motion is spherically symmetric, so that ρ depends only on the magnitude of $\boldsymbol{r}$. A spherical shell of unit surface density at $r = r_0$ contains

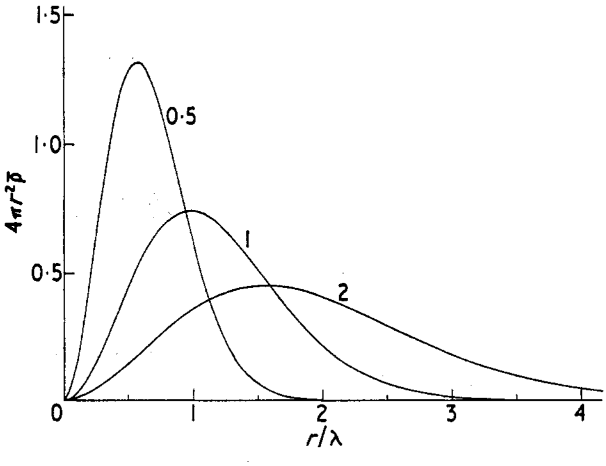

Figure 4. Spherical probability density $4\pi r^2\bar{\rho}(r; t)$ for electrons with a Maxwellian velocity distribution and constant mean free path when $\lambda^{-1}c_{\mathrm{rms}}t = 0{\cdot}5$, 1 and 2.

$4\pi r_0{}^2$ electrons and contributes a rectangular distribution to $W(x; t)$, that is

$$W(x; t) = 2\pi r_0 \text{ if } |x| \leqslant r_0.$$

Generalizing to any distribution $\rho(r;t)$, the surface density of a shell will be $\rho(r;t)\,\mathrm{d}r$, so that

$$W(x;t) = \int_x^\infty 2\pi r\rho(r;t)\,\mathrm{d}r. \tag{14}$$

Inverting this equation, we have

$$\begin{aligned}\rho(r;t) &= \frac{-1}{2\pi r}\left(\frac{\partial W}{\partial x}\right)_{x=r}\\ &= (2\pi^2 r)^{-1}\int_0^\infty \chi(k,t)k\sin kr\,\mathrm{d}k.\end{aligned} \tag{15}$$

This distribution contains a δ function at $r = ct$, consisting of those electrons which have not yet collided with a gas atom; those which do collide are left behind this expanding spherical front. The function $\bar{\rho}(r;t)$ is defined in a similar way and the probability density $4\pi r^2\bar{\rho}$ of the net displacement r in a time t is shown in figure 4.

4. Mean square displacement

Although all the moments of $W(x;t)$ can be computed numerically, an analytic expression for the mean value of x^2 as a function of time can be obtained from S. Differentiating equation (3) for S twice with respect to k, we have

$$\frac{\partial^2 S}{\partial k^2} = \iint -x^2\exp(\mathrm{i}kx-\mathrm{i}\omega t)W(x;t)\,\mathrm{d}x\,\mathrm{d}t.$$

If we set $k = 0$ and invert the Laplace transformation,

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\partial^2 S}{\partial k^2}\right)_{k=0}\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega &= \int_{-\infty}^{\infty} -x^2 W(x;t)\,\mathrm{d}x\\ &= -\langle x^2\rangle.\end{aligned} \tag{16}$$

Here we have

$$\frac{\partial^2 S}{\partial k^2} = \frac{2\gamma}{(1-\gamma S_0)^3}\left(\frac{\partial S_0}{\partial k}\right)^2 + \frac{1}{(1-\gamma S_0)^2}\frac{\partial^2 S_0}{\partial k^2}.$$

As k approaches zero, S_0 and its derivatives take limiting values:

$$\begin{aligned}S_0 &= \int (\gamma+\mathrm{i}\omega-\mathrm{i}kv)^{-1}f(v|c)\,\mathrm{d}v \to (\gamma+\mathrm{i}\omega)^{-1}\\ \frac{\partial S_0}{\partial k} &= \int (\gamma+\mathrm{i}\omega-\mathrm{i}kv)^{-2}\mathrm{i}vf(v|c)\,\mathrm{d}v \to \mathrm{i}\langle v\rangle(\gamma+\mathrm{i}\omega)^{-2}\\ \frac{\partial^2 S_0}{\partial k^2} &= \int (\gamma+\mathrm{i}\omega-\mathrm{i}kv)^{-3}-2v^2f(v|c)\,\mathrm{d}v \to -2\langle v^2\rangle(\gamma+\mathrm{i}\omega)^{-3}.\end{aligned}$$

The velocity distribution is symmetric, so that $\langle v\rangle$ and hence $\langle x(t)\rangle$ are zero. For the rectangular distribution $\langle v^2\rangle = \frac{1}{3}c^2$; thus

$$\langle x^2\rangle = \frac{c^2}{3\pi}\int_{-\infty}^{\infty}\frac{\mathrm{i}\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega}{\omega^2(\omega-\mathrm{i}\gamma)}.$$

This integral can be calculated by completing the contour in the upper half-plane, encircling the simple pole at $\omega = \mathrm{i}\gamma$ and the double pole at $\omega = 0$. The result is

$$\langle x^2\rangle = \tfrac{2}{3}c^2\gamma^{-2}(\gamma t - 1 + \mathrm{e}^{-\gamma t}). \tag{17}$$

For a Maxwellian distribution of electrons we have $\langle c^2 \rangle = 3\kappa T/m$. If there is a constant mean free path,

$$\langle x^2 \rangle = \int \tfrac{2}{3}\lambda^2 \left\{\frac{ct}{\lambda} - 1 + \exp\left(\frac{-ct}{\lambda}\right)\right\} f_0(c)\, \mathrm{d}c$$

$$= \tfrac{2}{3}\lambda^2\{\tfrac{1}{2}\bar{\gamma}t - 1 + (1+2\eta^2)\exp(\eta^2)\operatorname{erfc}(\eta)\} \tag{18}$$

where the average collision rate $\bar{\gamma} = (8\kappa T/\pi m)^{1/2}\lambda^{-1}$, $\eta = \frac{1}{4}\pi^{1/2}\bar{\gamma}t$ and the complementary error function

$$\operatorname{erfc}(\eta) = 2\pi^{-1/2}\int_\eta^\infty \exp(-y^2)\,\mathrm{d}y.$$

If γ is constant,

$$\langle x^2 \rangle = 2D\gamma^{-1}(\gamma t - 1 + \mathrm{e}^{-\gamma t}) \tag{19}$$

where the diffusion constant $D = \kappa T/m\gamma$. These expressions both tend towards $\langle x^2 \rangle = \langle v^2 \rangle t^2$ when t is small and towards $\langle x^2 \rangle = 2Dt$ when t is large, although, for equation (18), D takes the value $8\kappa T/3\pi m\bar{\gamma}$. If the density of gas atoms is adjusted to equalize these two diffusion constants, then at intermediate times the predictions for $\langle x^2 \rangle$ never differ by more than 5%. Thus the dependence of γ on speed has very little effect on the mean square displacement.

In comparing these results with that obtained by Ornstein for classical Brownian motion (equation (2)), the parameters β and γ may be equated as each represents the average rate of loss of ordered momentum. Thus equations (2) and (19) are found to be identical, even though in classical Brownian motion remanence of velocity is virtually complete, whereas here each impact completely destroys the previous velocity. The reason why equation (18) is not in exact agreement is that the electronic Brownian motion theory is not restricted by the requirement that β be independent of velocity. For example, when λ is constant $\beta = c/\lambda$, and c is obviously correlated with $|v|$.

5. Brownian motion in a magnetic field

When a magnetic field $\boldsymbol{B}$ is present, the paths of the electrons between collisions are helices about the field lines. If the angle between $\boldsymbol{k}$ and $\boldsymbol{B}$ is θ, then the $\boldsymbol{k}$ component of the displacement during one free path is

$$x = (c^2 - v^2)^{1/2}\omega_c^{-1}\sin\theta\{\sin(\omega_c t + \psi) - \sin\psi\} + vt\cos\theta.$$

Here v is the component of velocity parallel to $\boldsymbol{B}$, the cyclotron frequency is $\omega_c = eB/m$ and ψ is the azimuthal angle describing the initial direction of motion. The transform of this displacement is

$$S_0 = \int_0^\infty \mathrm{d}t \int_{-\infty}^\infty \mathrm{d}v \int_0^{2\pi} \frac{\mathrm{d}\psi}{2\pi} \exp\{\mathrm{i}z\sin(\omega_c t + \psi) - \mathrm{i}z\sin\psi + \mathrm{i}kvt\cos\theta - \mathrm{i}\omega t - \gamma t\} f(v|c)$$

where $z = k(c^2 - v^2)^{1/2}\omega_c^{-1}\sin\theta$. Using the Bessel function expansion formula

$$\exp(\mathrm{i}z\sin\psi) = \sum_{p=-\infty}^{\infty} \mathrm{J}_p(z)\exp(\mathrm{i}p\psi)$$

the integrand becomes

$$\sum_{pp'} \mathrm{J}_p(z)\mathrm{J}_{p'}(z)\exp\{\mathrm{i}p\omega_c t + \mathrm{i}(p-p')\psi + \mathrm{i}kvt\cos\theta - \mathrm{i}\omega t - \gamma t\}.$$

The integral over ψ is zero unless $p = p'$. Performing the t integration yields

$$S_0(k,\omega) = \sum_p \int \frac{\mathrm{J}_p{}^2(z) f(v|c)\,\mathrm{d}v}{\gamma + \mathrm{i}\omega - \mathrm{i}p\omega_c - \mathrm{i}kv\cos\theta}. \tag{20}$$

This expression can be evaluated numerically and the spectrum S obtained using equation (4). The spectrum exhibits peaks at the multiples of the cyclotron frequency as long as γ is not too large or θ too small. Figure 5 shows a few examples.

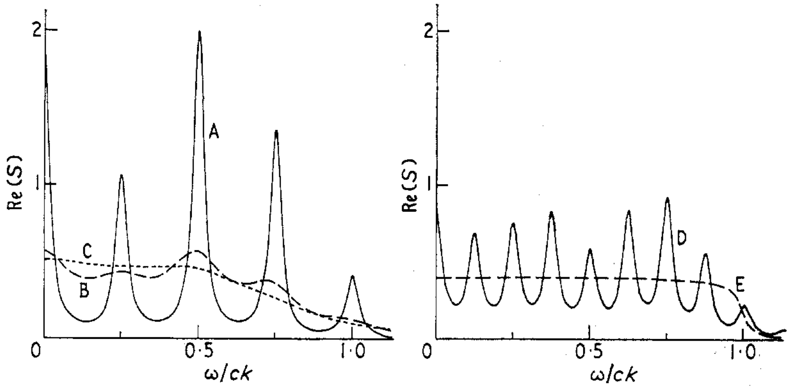

Figure 5. Spectra $S(k, \omega)$ for monoenergetic electrons in a magnetic field which is perpendicular to the wave vector k. The values of λk and ω_c/ck are respectively 40, 0·25 (A); 8, 0·25 (B); 4, 0·25 (C); 40, 0·125 (D); 40, 0 (E).

The mean square displacement in a magnetic field is calculated as in the previous section. The algebra is tedious, so that only the essential stages will be indicated below. The angle θ is taken to be $\frac{1}{2}\pi$ because the components of $\langle x^2 \rangle$ parallel and perpendicular to $\boldsymbol{B}$ can be calculated separately and compounded by Pythagoras's rule. Most of the Bessel functions and their first and second derivatives tend to zero with $\boldsymbol{k}$. The exceptions are

$$J_0^2 \to 1$$

$$\frac{d^2 J_0^2}{dz^2} \to -1$$

$$\frac{d^2 J_{\pm 1}^2}{dz^2} \to \tfrac{1}{2}.$$

Hence the limiting values of S_0 and its derivatives are

$$S_0 \to (\gamma + i\omega)^{-1}$$

$$\frac{\partial S_0}{\partial k} \to 0$$

$$\frac{\partial^2 S_0}{\partial k^2} \to \tfrac{2}{3}c^2 \omega_c^{-2}\{\tfrac{1}{2}(\gamma + i\omega - i\omega_c)^{-1} - (\gamma + i\omega)^{-1} + \tfrac{1}{2}(\gamma + i\omega + i\omega_c)^{-1}\}.$$

The integrand for $\langle x^2 \rangle$ has simple poles at $\omega = \pm\omega_c + i\gamma$ and a double pole at $\omega = 0$. The result of the contour integration is

$$\langle x^2 \rangle = \tfrac{2}{3}c^2 \left\{ \frac{\gamma t}{\gamma^2 + \omega_c^2} + \frac{(\omega_c^2 - \gamma^2)(1 - e^{-\gamma t} \cos \omega_c t) - 2\gamma\omega_c e^{-\gamma t} \sin \omega_c t}{(\gamma^2 + \omega_c^2)^2} \right\}. \tag{21}$$

The value of $\langle x^2 \rangle$ for a Maxwellian distribution of electrons with constant collision rate is obtained by replacing $\frac{1}{3}c^2$ by $\kappa T/m$ in equation (21). The total mean square displacement for large t in the plane perpendicular to $\boldsymbol{B}$ agrees with the well-known result (Allis 1956) $2D_\perp t$, where

$$D_\perp = \frac{2\kappa T}{m} \frac{\gamma}{\gamma^2 + \omega_c^2}. \tag{22}$$

If the collision frequency is low enough, then $\langle x^2 \rangle$ displays oscillations instead of increasing monotonically with time (see figure 6). For a fixed value of $\omega_c t$, choosing γ equal to ω_c provides the quickest possible diffusion. The curves are plotted to display this feature. On the other hand, for a fixed value of γt, the mean square displacement is greatest

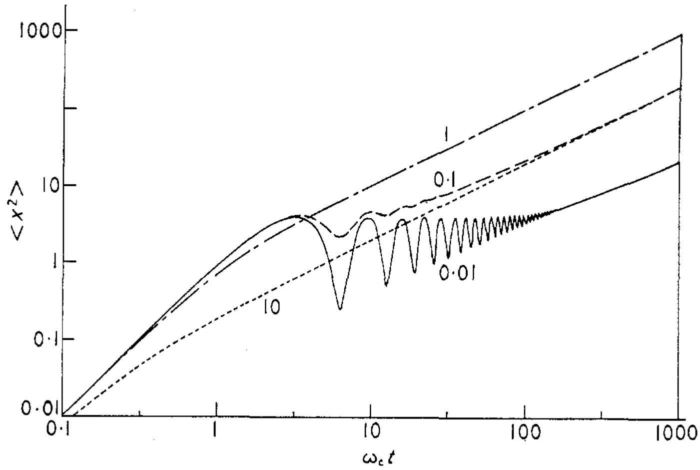

Figure 6. Mean square displacement of electrons perpendicular to a magnetic field. The ratios of the collision rate to the cyclotron frequency are 0·01, 0·1, 1 and 10.

when the magnetic field is turned off. When the electrons move with constant mean free path, integration of equation (21) does not yield a simple result, except when t is large. The motion is then characterized by

$$D_{\perp} = \tfrac{2}{3}\lambda^2 \bar{\gamma} t\{1-\zeta-\zeta^2 \,\mathrm{e}^{\zeta}\mathrm{Ei}(-\zeta)\} \tag{23}$$

where $\zeta = \omega_c{}^2\lambda^2 m/2\kappa T$ and the exponential integral

$$\mathrm{Ei}(-\zeta) = \int_{-\infty}^{\zeta} y^{-1}\,\mathrm{e}^{y}\,\mathrm{d}y.$$

6. Scattering of electromagnetic radiation

A free electron performing Brownian motion in a gas will scatter electromagnetic radiation incident upon it (Thomson scattering). If the incident plane wave has angular frequency Ω and wave vector $\boldsymbol{K}_0$, the amplitude of the spherical wave scattered through an angle θ with wave vector $\boldsymbol{K}_1$ is proportional to

$$A(k, t) = \cos\{\Omega t + \boldsymbol{k}\,.\,\boldsymbol{x}(t) + \phi\} \tag{24}$$

where $\boldsymbol{k} = \boldsymbol{K}_1 - \boldsymbol{K}_0$; hence $k \simeq 2K_0 \sin \frac{1}{2}\theta$. At time $t = 0$, x is defined to be zero, so that the phase ϕ contains a contribution depending on the initial position of the electron. The incident monochromatic wave is phase-modulated by movement of the scattering electron, and if there are no collisions a Doppler shift occurs, given by

$$A(k, t) = \cos\{(\Omega + \boldsymbol{k}\,.\,\boldsymbol{v})t + \phi\}.$$

The autocorrelation function for a group of non-interacting electrons is given, in the general case, by an ensemble average (Rice 1964):

$$\begin{aligned} C(k, t) &= \langle A(k, t)A(k, 0)\rangle \\ &= \tfrac{1}{2}\langle \cos(\Omega t + kx) + \cos(\Omega t + kx + 2\phi)\rangle. \end{aligned}$$

The second term here is zero because ϕ is randomly distributed. Thus

$$C(k, t) = \tfrac{1}{2} \int \bar{W}(x;\, t) \cos(\Omega t + kx)\, \mathrm{d}x.$$

The power spectrum for a wave observed at frequency $\Omega + \omega$ is (6):

$$P(k, \omega) = 4 \int_0^{\infty} C(k, t) \cos(\Omega t + \omega t)\, \mathrm{d}t.$$

If we ignore the cross term between positive and negative frequencies, which is valid if $\Omega \gg \omega$,

$$P(k, \omega) = \int_0^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}x \bar{W}(x;\, t) \cos(\omega t - kx)$$
$$= \mathrm{Re}\{\bar{S}(k, \omega)\}. \tag{25}$$

Figures 1, 2 and 5 show power spectra for the scattered radiation, and it is observed that, in the limit of $\lambda \to \infty$, the electron velocity distribution leads to Doppler broadening of the incident spectral line. When collisions occur, the broadening is reduced (Fejer 1960) because the Doppler effect depends on the average velocity over a finite time interval and collisions hinder the motion of the electrons. In atomic physics, collisions broaden the lines emitted by radiating atoms because the impacts cause abrupt random phase changes in the wave. Thus there is no correlation between the signals emitted before and after any collision and the reduced correlation time implies that the spectral line is broadened. On the other hand, when radiation is scattered by electrons, the phase of the scattered wave cannot change discontinuously because this would imply an instantaneous change of the electron's position. There is a 50% chance that the velocity of the electron changes sign at the collision, hence reversing the direction in which the phase modulation is changing and increasing the correlation time. In the limit of $\lambda \to 0$, the electron can hardly move, so that the correlation time becomes very long. The major effect of the collisions is this spectral narrowing, although the extreme wings of the spectrum are widened slightly. If the electron bounces back from an atom, the wave has a kink which corresponds to a modulating component at a high frequency.

7. Spectra with collective effects

The dressed test particle concept (Rosenbluth and Rostoker 1962) can be used to account for the binary correlations between electrons caused by their Coulomb interactions. Large-angle Coloumb scattering is ignored and it is assumed that the field particles are sufficiently numerous for each to suffer only a small perturbation. Thus one electron is singled out as the test particle to move unperturbed through the plasma, perturbing the continuous distribution of field particles as it goes. These perturbations scatter light coherently with that scattered by the test particle itself. Each electron in turn is considered as the test particle and the contributions from each are summed incoherently. The power spectrum can be written as

$$P(k, \omega) = \frac{\mathrm{Re}(\bar{S})}{|\epsilon|^2} \tag{26}$$

where $\epsilon(\boldsymbol{k}, \omega)$ is the dielectric constant of the plasma. Collisions of test particles with gas atoms have been dealt with in the earlier sections; the purpose here is to calculate ϵ, which describes the effect of field particle collisions.

The interaction between electrons and ions also result in collective effects but, because of the slow movement of the ions, only the scattering with very small ω is affected. This region will not be considered in this paper as the collisions between ions and gas atoms are not adequately described by either classical Brownian motion or the Lorentz gas theory presented here. Large-angle electron–electron and all electron–ion collisions are also ignored for the same reason. Thus the transition to acoustic waves at high pressures (Sitenko and Gurin 1966) is not obtained with this theory.

In setting up the linearized Boltzmann equation

$$\left(\frac{\partial}{\partial t}+\boldsymbol{v}_0\frac{\partial}{\partial \boldsymbol{x}}\right)f_1 = \frac{-e}{m}\nabla\Phi\,.\,\frac{\partial f_0}{\partial \boldsymbol{v}}+\left(\frac{\partial f_1}{\partial t}\right)_{\mathrm{coll}} \tag{27}$$

we express the deviation of the field particle distribution function from the equilibrium Maxwellian distribution $f_0(\boldsymbol{v})$ as an explicit function of speed, $f_1(\boldsymbol{x}, t, v, c)$. This is because the collisions do not intermingle the classes of field particles with different values of c. Here v denotes the component of $\boldsymbol{v}$ which is parallel to the wave vector $\boldsymbol{k}$. All quantities are independent of the third component of $\boldsymbol{v}$, so that this is not included in the arguments. The collision term

$$\left(\frac{\partial f_1}{\partial t}\right)_{\mathrm{coll}} = -\gamma(c)f_1+\gamma(c)f(v|c)\int f_1(v')\,\mathrm{d}v' \tag{28}$$

conserves particles in the proper way. Like the widely used term of Bhatnagar *et al* (1954, to be referred to as BGK)

$$\left(\frac{\partial f_1}{\partial t}\right)_{\mathrm{coll}} = -\gamma f_1+\gamma f_0(\boldsymbol{v})\int f_1(\boldsymbol{v}')\,\mathrm{d}\boldsymbol{v}' \tag{29}$$

it assumes isotropic scattering and no remanence of velocity. However, the velocity after a collision belongs to a rectangular distribution rather than a physically unrealistic Maxwellian distribution. The BGK term seems always to be used with constant γ (although a slightly less restrictive assumption is possible), but here any dependence of γ on c is allowed.

The acceleration term can change the speed of a particle, but we must evaluate the derivative at the given value of c:

$$\frac{\partial f_0}{\partial \boldsymbol{v}} = \frac{-\boldsymbol{v}m}{\kappa T}f_0(\boldsymbol{v}) = \frac{-\boldsymbol{v}m}{\kappa T}f(v|c)f_0(c).$$

Poisson's equation for the potential when the test particle's position is $\boldsymbol{x}'(t)$ is

$$\nabla^2\Phi = 4\pi e\delta\{\boldsymbol{x}-\boldsymbol{x}'(t)\}+4\pi n_{\mathrm{e}}e\int f_1\,\mathrm{d}\boldsymbol{v} \tag{30}$$

where n_{e} is the average density of electrons. Looking for solutions in the form of plane waves, we resolve

$$p(k,\omega) = (2\pi)^{-4}\iint \delta\{\boldsymbol{x}-\boldsymbol{x}'(t)\}\exp(-\mathrm{i}\boldsymbol{k}\,.\,\boldsymbol{x}+\mathrm{i}\omega t)\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\omega$$

with similar definitions for the transforms $f_1(\boldsymbol{k}, \omega, v, c)$ and $\Phi(\boldsymbol{k}, \omega)$. The dielectric constant $\epsilon(\boldsymbol{k}, \omega)$ is defined by

$$\frac{p}{\epsilon} = p+n_{\mathrm{e}}\int f_1\,\mathrm{d}\boldsymbol{v}. \tag{31}$$

The equations (27) and (30) reduce to

$$(\mathrm{i}\omega-\mathrm{i}\boldsymbol{k}\,.\,\boldsymbol{v})f_1 = \frac{-e}{m}\Phi\frac{\mathrm{i}\boldsymbol{k}\,.\,\boldsymbol{v}m}{\kappa T}f(v|c)f_0(c)-\gamma f_1+\gamma f(v|c)\int f_1\,\mathrm{d}v' \tag{32}$$

and

$$-k^2\Phi = \frac{4\pi e p}{\epsilon}. \tag{33}$$

If we introduce the dimensionless parameter (Salpeter 1960)

$$\alpha^2 = \frac{4\pi n_e e^2}{\kappa T k^2} = \frac{1}{l_D{}^2 k^2} \tag{34}$$

where l_D is the electronic Debye length and writing $a = \alpha^2 p/\epsilon n_e$, these equations yield

$$(\gamma + i\omega - i\boldsymbol{k}\,.\,\boldsymbol{v})f_1 = ai\boldsymbol{k}\,.\,\boldsymbol{v}f(v|c)f_0(c) + \gamma f(v|c)\int f_1\,dv'.$$

Therefore

$$\int f_1\,dv = af_0(c)\int\frac{i\boldsymbol{k}\,.\,\boldsymbol{v}f(v|c)\,dv}{\gamma+i\omega-i\boldsymbol{k}\,.\,\boldsymbol{v}} + \gamma\int f_1\,dv'\int\frac{f(v|c)\,dv}{\gamma+i\omega-i\boldsymbol{k}\,.\,\boldsymbol{v}}$$

$$= af_0(c)\{(\gamma+i\omega)S_0 - 1\} + \gamma S_0\int f_1\,dv'$$

$$= af_0(c)(i\omega S - 1).$$

Integration over the speed gives

$$\iint f_1\,dv\,dc = a(i\omega\bar{S} - 1)$$

hence

$$\frac{p}{\epsilon} = p + \frac{\alpha^2 p}{\epsilon}(i\omega\bar{S} - 1)$$

and finally

$$\epsilon = 1 - \alpha^2(i\omega\bar{S} - 1). \tag{35}$$

In the limit of $\gamma \to 0$,

$$S \to S_0$$

$$\bar{S} \to \iint f_0(c)f(v|c)(\gamma+i\omega-i\boldsymbol{k}\,.\,\boldsymbol{v})^{-1}\,dv\,dc = \int f_0(\boldsymbol{v})(\gamma+i\omega-i\boldsymbol{k}\,.\,\boldsymbol{v})^{-1}\,d\boldsymbol{v}$$

and

$$\epsilon \to 1 - \alpha^2\int i\boldsymbol{k}\,.\,\boldsymbol{v}f_0(\boldsymbol{v})(\gamma+i\omega-i\boldsymbol{k}\,.\,\boldsymbol{v})^{-1}\,d\boldsymbol{v}$$

in agreement with the results of Salpeter (1960), Rosenbluth and Rostoker (1962) and others. Figure 7 shows the power spectra for various values of λk and α. If α is high enough for a

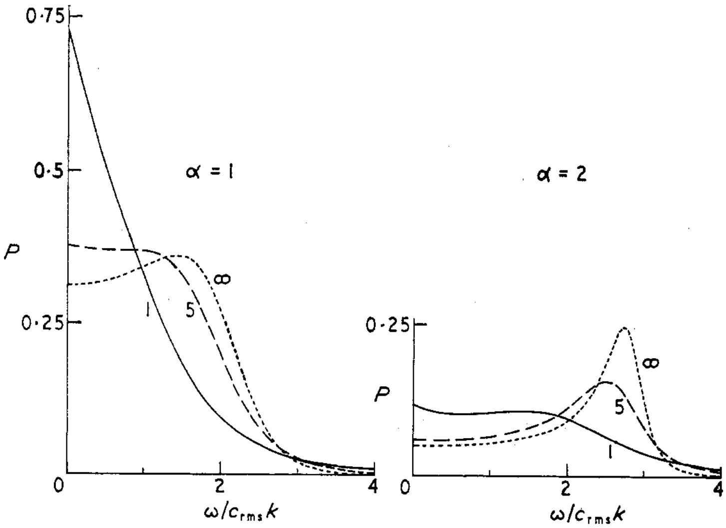

Figure 7. Laser scattering spectra $P(k, \omega)$ from a plasma with collective effects characterized by $\alpha = 1$ and 2 and $\lambda k = 1$, 5 and ∞.

distinct satellite near the electron plasma frequency ω_p, collisions broaden this peak and shift it towards the centre frequency. Simple theories which merely add a collisional damping term to that caused by Landau damping do not predict this shift.

It can readily be seen that deduction of plasma parameters like n_e and T_e from the observed spectra can be invalidated by collisions between electrons and neutral particles. *A fortiori* the same comment applies to sophisticated attempts (Brown and Rose 1966) to deduce the whole velocity distribution from the results of scattering experiments.

8. Comparison with other work

Fejer (1960) used classical Brownian theory in the diffusion limit and found that the asymptotic form of the spectrum was Lorentzian with width Dk^2. A similar but not identical result can be obtained here by approximating equation (5). When the collision rate is very much larger than both ω and ck, the logarithms can be expanded to give

$$S_0 \simeq (\gamma + \mathrm{i}\omega)^{-1}\left(1 - \frac{\frac{1}{3}c^2k^2}{\gamma^2}\right)^{-1}$$

therefore

$$S \simeq \left(\mathrm{i}\omega + \frac{\frac{1}{3}c^2k^2}{\gamma}\right)^{-1}.$$

Thus the central part of the spectrum for monoenergetic electrons is approximately Lorentzian with width $\frac{1}{3}c^2k^2/\gamma$. Although at first sight it appears that averaging over the speed c would give Fejer's result, this is not correct. $\bar{S}(k, \omega)$ can maintain the Lorentzian character only in the unlikely event of γ being proportional to c^2. Fejer's result corresponds to averaging merely the width, not the spectrum itself.

The first calculations (Hagfors 1961, Renau *et al.* 1961), which include both collective effects and collisions, used as collision term in the Boltzmann equation

$$\left(\frac{\partial f_1}{\partial t}\right)_{\mathrm{coll}} = -\gamma(f - f_0) = -\gamma f_1. \tag{36}$$

This means that the collisions distribute the electrons uniformly in real space as well as in velocity space, so that each perturbation f_1 is annihilated as soon as the electron carrying it collides with another particle. The results of this assumption are obtained from the collisionless theory by replacing ω by $\omega - \mathrm{i}\gamma$. A more detailed criticism of this work is given by Dougherty and Farley (1963), who themselves use the BGK term (equation (29)). However, this is also unsatisfactory as it distributes the electrons in energy space, in addition to merely changing the direction of motion. In the notation of this paper they define $\bar{S}$ as $\bar{S}_0(1 - \gamma\bar{S}_0)^{-1}$, instead of calculating S and then applying the average over c. Dougherty and Farley give only a brief description of the effect of collisions on the satellite peak (their main interest is in the ion peak). They state that the general effect is to replace ω by $\omega - \mathrm{i}\gamma$, but in fact their theory predicts a shift towards $\omega = 0$, as does the electronic Brownian motion theory developed here. Similar results have been obtained by Lewis and Keller (1962) and Taylor and Comisar (1963).

Grewal (1964 a) has studied the effect of collisions in a plasma using the Fokker–Planck equation, which Chandrasekhar (1943) derived for classical Brownian motion. The collision term in the linearized equation is

$$\left(\frac{\partial f_1}{\partial t}\right)_{\mathrm{coll}} = \beta\frac{\partial}{\partial \boldsymbol{v}}\cdot\left(\boldsymbol{v} + \frac{\kappa T}{m}\frac{\partial}{\partial \boldsymbol{v}}\right)f_1. \tag{37}$$

He used this expression for both ions and electrons and obtained spectral narrowing if $\alpha \ll 1$. When α is large he calculated the form of the ion peak near $\omega = 0$, but stated that the satellite peak near ω_p is not described adequately by his theory. In a later paper

(Grewal 1964 b) he compared his results with those of Dougherty and Farley (1963) and found that the probability densities only agree at large t, although for a Maxwellian distribution of velocities the discrepancies at earlier times are not large. However, his approximation for $W_{FP}(x; t)$ when $\beta t \ll 1$ is incorrect. He assumes a Gaussian distribution with $\langle x^2 \rangle = (\kappa T/m)t^2(1+\frac{2}{3}\beta t)$, while the exact expression is Gaussian with $\langle x^2 \rangle$ given by equation (2). (This result can be obtained from equation (171) of Chandrasekhar's (1943) paper by integrating over two components of the displacement and averaging over the initial velocity.) At $\beta t = 0{\cdot}5$ his approximation is in error by more than the difference he shows between the FP and BGK results. The exact expression is compared in figure 3 with the density computed using the electronic Brownian motion theory for electrons having a constant mean free path. (The times for which the curves apply have been adjusted to give equal variances.) As can be seen, the electronic Brownian motion curves have slightly sharper peaks and longer tails than do pure Gaussian distributions.

A further comparison between the theories is provided by the spectra scattered by a Maxwellian distribution of electrons (see figure 8). The BGK theory predicts a smooth

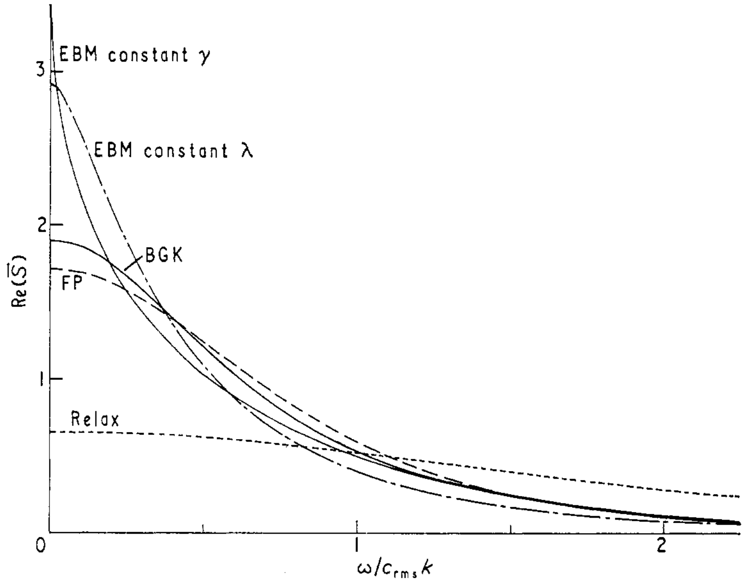

Figure 8. Spectra $\bar{S}(k, \omega)$ for electrons with a Maxwellian velocity distribution, comparing the electronic Brownian motion (EBM) theory with other theories. The wave number k is λ^{-1}, $c_{rms}\gamma^{-1}$ or $c_{rms}\beta^{-1}$.

profile, while the electronic Brownian motion spectra are more sharply peaked near $\omega = 0$. This occurs because the electrons are not all equivalent, but some diffuse more slowly than others, thereby giving more collisional narrowing of the spectra. This is particularly severe if γ is constant because then the slowest electrons are condemned to make tiny steps which are much smaller than k^{-1}. The relaxation spectrum $\bar{S}_0(k, \omega)$ corresponds to the collision term given in equation (36). It also applies if each electron is absorbed at its first collision, other electrons being released elsewhere in the plasma to keep the density constant. This spectrum is very broad because there is no mechanism whereby the correlation time can be increased. The Fokker–Planck spectrum was obtained by transforming $W_{FP}(x; t)$ to give

$$\chi_{FP}(k, t) = \exp\{-\tfrac{1}{2}k^2\langle x^2(t)\rangle\} \tag{38}$$

from which it is easily shown (Singwi and Sjölander 1960) that

$$S_{FP}(k, \omega) = \exp\left(\frac{k^2D}{\beta}\right)\sum_{n=0}^{\infty}\frac{(-k^2D/\beta)^n}{n!(i\omega+k^2D+n\beta)}. \tag{39}$$

This is very similar to the BGK profile; in both cases every electron ranges over all possible speeds, and it is seen to be unimportant whether the speed changes abruptly or gradually. Which of these five theories gives the best description of the scattering of radiation by electrons will have to be determined by experiment, but in the meantime it should be borne in mind that the electronic Brownian motion theories assume the electron to be infinitely lighter rather than infinitely heavier than the gas atoms.

Kurşunoğlu (1962) has applied classical Brownian theory to the motion of an electron in a magnetic field with a uniform positive background. There is a fluctuating electric field and a dynamical friction interpreted as the 'collision' of the electron with the 'oscillators' of the electromagnetic field. Using a method similar to Chandrasekhar's, he obtains a formal solution for $W(\boldsymbol{r}; t|\boldsymbol{v}_0)$ which can be written out explicitly only if t is large. The mean square displacements parallel and perpendicular to B then take their usual form (Allis 1956). In a later paper (Kurşunoglu 1963) he generalizes his work to allow an anisotropic dynamical friction, but he does not relate this to the properties of the plasma in any way.

Liboff (1966) has used classical Brownian theory to describe the motion of electrons in electric and magnetic fields. When $E = B = 0$, he finds that

$$\langle r^2 \rangle = 2D\beta^{-1}\left(\tfrac{1}{2}\beta t - 1 + \frac{1-e^{-\beta t}}{\beta t}\right) + \mathrm{O}\left(\frac{1}{\beta t}\right)$$

where D has its usual value $\kappa T/m\beta$. Because of the unknown term in $1/\beta t$, this formula is useless unless $\beta t \gg 1$. It is therefore not surprising that "although the asymptotic formula agrees with that of previous investigators, there is disagreement for earlier times". In fact, not even the asymptotic formula is in agreement because, as Liboff shows elsewhere in his paper, in the limit of $t \to \infty$ his formula gives only half the displacement predicted by Einstein. This arises because he defines a time average

$$\langle r^2 \rangle = t^{-1} \int_0^t r^2(t')\,\mathrm{d}t'$$

whereas the conventional definition uses an ensemble average in accordance with the kind of experiments which are actually performed (Golant 1963).

9. Application

The electronic Brownian motion theory should be particularly useful in describing a helium plasma. Most collisions are elastic because of the high excitation and ionization potentials, and the mean free path is constant (Golden 1966) within 10% for an electron energy below 3 ev. For an arc at atmospheric pressure in thermal equilibrium (Drawin and Felenbok 1965) at 1 ev, using light from a ruby laser with scattering angle 3° or less, electron–atom collisions cause most of the broadening of the satellite peak. Large-angle Coulomb collisions (Boyd *et al.* 1966) are next in importance. The plasmas which are used in magnetohydrodynamic studies are often seeded with alkali metals to increase the electron density at low temperatures. These metals furnish very large cross sections for electron collisions. The satellite peak from a caesium plasma at $\frac{1}{4}$ ev and a partial pressure of 0·01 atm observed at 10° is doubled in width by collisions. At 0·1 atm narrowing of the whole spectrum would be severe because $\lambda k = 1$.

The BGK collision model has been applied by Nelkin and Ghatak (1964) to the scattering of slow neutrons by a liquid. Wittke and Dicke (1956) have observed collisional narrowing of the microwave Doppler absorption profile of hydrogen atoms caused by the buffering action of molecular hydrogen. They calculated the asymptotic form of the spectrum, finding the Lorentzian shape later obtained by Fejer (1960) for electrons in a plasma. In these and similar experiments the electronic Brownian motion theory will give a better description if the scattering or absorbing particle is lighter than the fluid particles.

10. Conclusion

The theory of electronic Brownian motion describes the motion of a light particle in a fluid, by giving results such as probability densities and spectra. The description is at least as detailed as the classical results obtained for a heavy particle. For example, an exact expression is obtained for the mean square displacement in a magnetic field, whereas only the asymptotic value seems to have been derived using classical theory.

The theory of laser scattering experiments is extended to allow for realistic collisions of the electrons with neutral atoms. The observed spectra are altered drastically when these collisions are an important factor. There are qualitative differences between the predictions of the electronic Brownian motion theory on the one hand and the BGK and Fokker–Planck results on the other, which should allow an experimental choice to be made between the theories.

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† The references Chandrasekhar (1943), Rice (1944) and Uhlenbeck and Ornstein (1930) are reprinted in the publication by Wax (1954).